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Rational Approximation to e^{-x}

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Recently much attention has been paid to the problem of obtaining uniform approximations to e^{-x} by rational functions on the whole positive axis. It has been shown, for example, that one can achieve an error of roughly 3^{-n}, and not much better, by using reciprocals of polynomials of degree n, see [1] and [2].

One problem left open was whether one can achieve better than a c^n error by the use of general rational functions. We show, in this note, that one cannot.

THEOREM. *Let $P(x)$, $Q(x)$ be any polynomials of degree $<n$. There must be a point on the positive axis where*

$$\left| e^{-x} - \frac{P(x)}{Q(x)} \right| > 1280^{-n}.$$

Proof. The simple substitution of $x \log 2$ for x changes e^{-x} to 2^{-x} and this will prove more convenient for our calculations. We show, in fact, that $|2^{-x} - P(x)/Q(x)| > 1280^{-n}$ somewhere in $[0, 3n]$. Suppose, otherwise, that

$$|2^{-x}Q(x) - P(x)| \leqslant 1280^{-n} |Q(x)| \qquad \text{throughout } [0, 3n]. \tag{1}$$

Now normalize so that

$$\max_{[0,n]} |Q(x)| = 1. \tag{2}$$

It follows that the function $Q((n/2)t + n/2)$ is bounded by 1 on $[-1, 1]$ and is, therefore, bounded by the Tchebychev polynomial outside $[-1, 1]$. This tells us that, throughout $[0, 3n]$, $|Q(x)| \leqslant T_n(5) = \frac{1}{2}(5 + (24)^{1/2})^n + \frac{1}{2}(5 - (24)^{1/2})^n < 10^n$. If we insert this estimate into (1) we obtain

$$|2^{-x}Q(x) - P(x)| \leqslant 2^{-7n} \qquad \text{throughout } [0, 3n]. \tag{3}$$

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Next we introduce the difference operator Δ defined, as usual, by $\Delta f(x) = f(x+1) - f(x)$, and we recall the usual notations in the operator calculus, e.g. $(2\Delta^2 - 3)f(x)$ means $2\Delta(\Delta f(x)) - 3f(x)$, etc.

If we call $2^{-x}Q(x) - P(x) = R(x)$, and we observe that $\Delta^n P(x) = 0$ (because deg $P < n$) then we obtain

$$\Delta^n(2^{-x}Q(x)) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} R(x+j).$$

For x in $[0, 2n]$, moreover, (3) is applicable to each of the terms $R(x+j)$ and we obtain the estimate

$$|\Delta^n 2^{-x}Q(x)| \leqslant 2^{-7n} \sum_{j=0}^{n} \binom{n}{j} = 2^{-6n}.$$

Now we use the "shift rule" (or ordinary induction) to deduce the identity $\Delta^n 2^{-x}Q(x) = 2^{-(x+n)}(\Delta - 1)^n Q(x)$ and inserting this into the above estimate gives

$$|(1-\Delta)^n Q(x)| \leqslant 2^{x+n}2^{-6n} \leqslant 2^{-3n} \qquad \text{throughout } [0, 2n]. \tag{4}$$

At this point we call $(1-\Delta)^n Q(x) = S(x)$ so that $Q(x) = (1-\Delta)^{-n} S(x)$. The expansion

$$(1-\Delta)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \Delta^k$$

is valid when applied to polynomials and since deg $S < n$, we have, in fact,

$$Q(x) = \sum_{k=0}^{n} \binom{n+k-1}{k} \Delta^k S(x)$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n+k-1}{k}\binom{k}{j} (-1)^{k-j} S(x+j).$$

If, finally, we restrict x to lie in $[0, n]$ then all the terms $S(x+j)$ are subject to (4) and there results the estimate

$$|Q(x)| \leqslant 2^{-3n} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n+k-1}{k}\binom{k}{j} = 2^{-3n} \sum_{k=0}^{n} \binom{n+k-1}{k} 2^k$$

$$\leqslant 2^{-2n} \sum_{k=0}^{n} \binom{n+k-1}{k} = 2^{-2n} \binom{2n}{n} < 1.$$

This holding throughout $[0, n]$ flatly contradicts (2) and the proof is complete.

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